VOLUME 79 SEPARATE No. 334

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

NOVEMBER, 1953

LATERAL BUCKLING OF CHANNELS AND Z-BEAMS

by H. N. Hill, M. ASCE

ENGINEERING MECHANICS DIVISION

{Discussion open until March 1, 1954}

Copyright 1953 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings — Separate No. 290). For free copies of this Separate—describing style, content, and format—address the Manager, Technical Publications, ASCE.

Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

The Society is not responsible for any statement made or opinion expressed in its publications.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

LATERAL BUCKLING OF CHANNELS AND Z-BEAMS

By H. N. Hill,[1] M. ASCE

[1] Asst. Chf., Eng. Design Div., Aluminum Co. of America, New Kensington, Pa.

Synopsis

Beams having moments of inertia that differ when referred to the two principal axes may buckle in the direction of least stiffness when loaded in the direction of greater stiffness. The author presents the results of a series of investigations into the lateral buckling of channel sections and Z-sections that are loaded in the direction of greater stiffness. The investigations included tests on beams and an analytical consideration of the problem which was aimed at devising a rational procedure for the design of such beams. Because of the resistance to twist, arising from the restraint to warping of the cross sections of the beams, the torsion-bending factor is introduced. This factor is shown to be a property of the cross section as vital to proper design as the commonly used section modulus and moment of inertia.

Introduction

Notation.—The letter symbols introduced in this paper are defined where they first appear, in the text or illustration, and are assembled alphabetically in Appendix II for convenience of reference.

A beam which has greater moment of inertia about one principal axis than about the other principal axis is subject to failure, by buckling, in the direction of the least stiffness when loaded so as to produce bending in the direction of the greatest stiffness. Several experimental investigations have been conducted to study this lateral buckling as applied to deep rectangular beams[2] and I-beams.[3,4] The unsymmetrical nature of channel sections and Z-sections

[2] "The Lateral Instability of Deep Rectangular Beams," by C. Dumont and H. N. Hill, *Technical Note No. 601*, National Advisory Committee for Aeronautics, Washington, D. C., May, 1937.

[3] "The Lateral Stability of Equal-Flanged Aluminum Alloy I-Beams Subjected to Pure Bending," by C. Dumont and H. N. Hill, *Technical Note No. 770*, National Advisory Committee for Aeronautics, Washington, D. C., August, 1940.

[4] "The Lateral Instability of Unsymmetrical I-Beams," by H. N. Hill, *Journal of the Aeronautical Sciences*, March, 1942.

introduces considerations not encountered in rectangular beams and I-beams.

The investigation described herein is concerned with the lateral buckling of equal-flanged channel and Z-beams when subjected to pure bending in a plane parallel to the web. The investigation included tests on beams and an analytical consideration of the problem which was directed toward devising a rational procedure for the design of such beams.

The Tests

The specimens tested were extruded aluminum alloy 17S-T6. Cross sections of the beams are shown in Fig. 1. The maximum variation of the individual measurements from the average values was less than 1%. The average measured dimensions were used in computing the section elements shown in Table 1.

TABLE 1.—Values of the Section Elements for the Test Beams

Section element	Channel section	Z-section
I_x, moment of inertia about the x-axis, in.^{4}	0.8090	1.0221
I'_x, moment of inertia about the x'-axis, in.^{4}	0.6822	
I_y, moment of inertia about the y-axis, in.^{4}	0.0728	0.0903
I'_y, moment of inertia about the y'-axis, in.^{4}	0.1996	
J, torsion factor, in.^{4}	0.00936	0.00392
Z, section modulus, in.^{3}	0.5750	0.6317
C, torsion-bending factor, in.^{6}	0.1401	0.1565

The tensile properties, as determined from flange specimens of the different lengths of material from which the beams were cut, are listed in Table 2. The modulus values are typical for this alloy and were used in the computations involved in the investigation. The modulus of elasticity E was 10,500,000 lb per sq in. and the modulus of rigidity G was 3,900,000 lb per sq in.

TABLE 2.—Mechanical Properties of Test Material

Section	Unsupported beam length, in inches	Tensile strength, in pounds per square inch	Tensile yield strength,[a] in pounds per square inch	Elongation, percentage in 2 in.
Channel	65	62,770	44,900	17.0
Channel	45	56,070	39,900	24.5
Channel	30	62,770	44,900	17.0
Channel	20	57,110	39,700	25.0
Z	65	53,140	36,800	27.0
Z	45	53,010	36,300	25.0
Z	30	53,140	36,800	27.0
Z	20	53,190	36,400	24.0

[a] 0.2% offset.

In order to provide a stable test setup, the beams were tested in pairs and loaded so as to subject the unsupported length to pure bending. Fig. 2 shows the general arrangement of the test. For both the channel sections and the Z-sections, four pairs of beams, having laterally unsupported lengths of 65 in., 45 in., 30 in., and 20 in., were tested.

The tests were made in a 40,000-lb-capacity testing machine. The load

was applied in increments and continued to be applied until failure occurred. Strains were measured near midspan with tensometers, of 1-in. gage length, mounted at the toe and heel of both flanges of one beam of each pair. The tensometers were removed before failure.

Test Results

Figs. 3 and 4 show that there was good agreement between the computed stress values and the stresses measured on both flanges of one beam of each pair tested. This agreement indicates that both beams of the pair were equally loaded.

The difference in the stress measured at the toe and at the heel of the compressive (top) flanges is associated with the lateral deflection of these flanges resulting from a slight initial crookedness, and it does not indicate horizontal bending of the beams. This statement is supported by the fact

TABLE 3.—Experimental and Computed Values of the Critical Bending Moments and Stresses

Section	Unsupported beam length, in inches	Experimental		Computed			
				$K=1$[a]		$K=0.5$[b]	
		M_c[c]	s[d]	M_c[c]	s[d]	M_c[c]	s[d]
Channel	65	12,900	20,400	6,500	10,300	16,500	26,100
Channel	45	18,450	29,200	10,400	16,500	29,600	46,900
Channel	30	24,300[e]	38,400	18,600	29,400	60,200	95,400
Channel	20	25,950[e]	41,000	36,100	57,100	138,500	219,000
Z	65	10,710	18,600	7,800	13,600	17,600	30,600
Z	45	14,280	24,800	11,900	20,700	29,000	50,500
Z	30	17,790	30,900	19,500	33,900	54,000	94,000
Z	20	21,300	37,000	34,400	59,800	109,000	190,000

[a] When the fixity coefficient $K=1$, the ends of the unsupported lengths are simply supported against lateral bending. [b] When the fixity coefficient $K=0.5$, the ends of the unsupported lengths are fixed against lateral bending. [c] The critical bending moment, in inch-pounds. [d] The stress, in pounds per square inch, computed from $s=M_c/Z$. [e] Failure originated as crumpling of the compression flange.

that, in the tension (bottom) flange, no appreciable difference between the stress at the toe and the heel was encountered.

Because of the manner in which the moments were applied to the beams, the unsymmetrical character of the channel sections and Z-sections did not affect the stress distribution. The resulting stresses were those that could be computed by the beam formula, $s=M/Z$, in which s is the normal unit stress and M denotes the bending moment. The stress distribution was not affected in the case of the channel, since the bending moment was applied in a plane parallel to a principal plane and the plane of the bending moment can be considered as containing the shear center of the section. Thus, no twisting of the channel section would be anticipated. In the case of the Z-beams, since the external moment was applied in a plane parallel to the web, which is not a principal plane, it can be expected that the applied moment would have components in both principal planes and that there would be horizontal as

well as vertical bending, with a consequent unsymmetrical stress distribution. However, the bending moment was introduced at the end of the unsupported length in the form of uniform stresses in the flanges, and this stress condition must persist throughout the unsupported length. In order to attain this stress distribution, the bending moment in the plane of the web must be accompanied by a moment in a plane perpendicular to the web. These horizontal moments, acting on both beams, are self-balancing in the end frames.

The maximum bending moments and the corresponding modulus of failure values for the beams of various unsupported lengths are shown in Table 3. Failure occurred by a sudden buckling in all tests, accompanied by a "falling off" of the load. Only the beams with unsupported lengths of 65 in. failed in the elastic range. Failure of the beams of shorter length involved plastic action—probably only to a slight extent in the beams with 45-in. unsupported lengths. In the two shortest spans of the channel beams, failure occurred by crumpling of the compression flange followed by lateral buckling of the beams

Analysis of Test Results

J. N. Goodier obtained an equation for computing the critical bending moments for beams having equal-flanged I-sections, channels sections, or Z-sections when subjected to pure bending in the principal planes.[5] This equation is of the form:

[5] "Flexural-Torsional Buckling of Bars of Open Sections," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., January, 1942.

$$P_n M^2_z + Q_n M^2_y - P_n Q_n T_n r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

in which M_z is the bending moment in the plane normal to the y-axis; M_y denotes the bending moment in the plane normal to the z-axis; r is the radius of gyration of the section about an axis through the shear center and normal to the x-axis and y-axis;

$$P_n = E I_y \left(\frac{\pi}{K L}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

$$Q_n = E I_z \left(\frac{\pi}{K L}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1c)$$

and

$$r^2 T_n = E C_s \left(\frac{\pi}{K L}\right)^2 + G J \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1d)$$

in which L is the unsupported length.

For the equal-flanged channel section, with bending applied in the plane of the web, $M_z = 0$; and the equation for the critical bending moment can be written as

$$M_c = \frac{\pi}{K L} \sqrt{\left[G J + \left(\frac{\pi}{K L}\right)^2 E C_s\right] E I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In the tests on the Z-beams, bending moments were applied parallel to and normal to the web, rather than in the principal planes. The moment

normal to the web had a definite relationship to the applied moment, M, in the plane of the web because the restraining frames prevented horizontal deflection of the beams (except as related to lateral buckling). The relative magnitude of the horizontal bending moment, and the values for bending moments in the principal planes of the **Z**-section corresponding to the applied moments, are investigated in Appendix I. The equations derived are

$$M_y = 1.079\ M \quad \text{.....(3}a\text{)}$$

and

$$M_z = 0.044\ M \quad \text{.....(3}b\text{)}$$

Substitution of Eqs. 3 into Eq. 1*a* yields, for the critical bending moment of the **Z**-beams loaded as described previously,

$$M_c = 0.927\ \frac{\pi}{K\,L}\sqrt{\left[G\,J + \left(\frac{\pi}{K\,L}\right)^2 E\,C_s\right] E\,I_y} \quad \text{.....(4)}$$

The term C_s is introduced into Eqs. 2 and 4 because of the additional resistance to twist arising from the restraint to warping of the cross sections of the beam.[6,7,8,9] For channel sections, C_s can be expressed as follows:

[6] "Torsion and Buckling of Open Sections," by H. Wagner, *Technical Memorandum No. 807*. National Advisory Committee for Aeronautics, Washington, D. C., 1936.

[7] "Torsion and Buckling of Open Sections," by H. Wagner and W. Pretschner, *Technical Memorandum No. 784*, National Advisory Committee for Aeronautics, Washington, D. C., 1936.

[8] "Twisting Failure of Centrally Loaded Open-Section Columns in the Elastic Range," by R. Kappus, *Technical Memorandum No. 851*, National Advisory Committee for Aeronautics, Washington, D. C., 1938.

[9] "Torsional and Flexural Buckling of Bars of Thin-Walled Open Section under Compressive and Bending Loads," by J. N. Goodier, *Journal of Applied Mechanics*, September, 1942.

$$C_s = \frac{I_f\,h^2}{2}\left(4 - \frac{18}{\frac{A_w}{A_f} + 6}\right) \quad \text{.....(5)}$$

and for **Z**-sections, C_s can be expressed as

$$C_s = \frac{I_f\,h^2}{2}\,4 - 6\,\frac{A_f}{A} \quad \text{.....(6)}$$

in which I_f is the moment of inertia of one flange about an axis through the centroid of the flange and parallel to the web, h denotes the height of the web, A_w is the area of the web, A_f is the area of one flange, and A is the total cross-sectional area.[10]

[10] "Torsion of Flanged Members with Cross Sections Restrained Against Warping," by H. N. Hill, *Technical Note No. 888*, National Advisory Committee for Aeronautics, Washington, D. C., March, 1943.

Substitution of the dimensions of the sections into Eqs. 5 and 6 yields the values for the torsion-bending factor listed in Table 1. In computing the values for the torsion factor J given in Table 1, it was assumed that the torsion factor for the section was equal to the sum of the torsion factors for the rectangles comprising the section. That part of the section common to the web and flange was considered in both the web and flange rectangles.

The derivation of Eqs. 2 and 4 is based on the assumption that the material is elastic. In Table 3 a comparison can be seen between the critical bending moments determined from the tests and corresponding values computed according to Eqs. 2 and 4. Computed stress values corresponding to the

critical bending moment values are also listed. Only the values above the horizontal lines in Table 3 are within the region of elastic action. The experimental stress values and the computed bending moment and stress values below these lines, therefore, do not represent true values. The values listed as experimental stress values are modulus of failure values for the beams.

Only the beams with unsupported lengths of 65 in. failed within the elastic range (Table 3). The critical bending moment values determined from the tests on these beams are between the corresponding computed values for simply supported and fixed-end conditions. The degree of end fixity obtained in the tests can be evaluated by substituting the experimentally determined values of critical bending moment into Eqs. 2 and 4 and solving for the fixity factor K. Accordingly, the fixity factors for the tests on the beams with unsupported lengths of 65 in. are, for the channels, $K = 0.59$ and, for the **Z**-beams, $K = 0.76$. Complete fixity was not attained in either test. The ends of the unsupported lengths of the channel beams were not as completely fixed as were the **I**-beams in tests previously reported.[3] This was not unexpected, since in the **I**-beam tests both flanges were tightly fitted into slots machined into the top and bottom plates, of the end frames, and, in the tests reported on herein, fixity depended on bolted connections. The K-value for the **Z**-beam is appreciably greater than 0.5, since the restraint is supplied for bending normal to the web. However, buckling occurs in the principal plane of least stiffness, which is at an angle of 24.526° to the plane of maximum restraint.

For design purposes, it is convenient to solve problems involving buckling beyond the region of elastic action by the equivalent slenderness-ratio method.[11]

[11] *Alcoa Structural Handbook*, Aluminum Co. of America, Pittsburgh, Pa., 1948.

In this method the expression for the critical stress is equated to the expression for the critical stress for a column $\frac{\pi^2 E}{(K L/r)^2}$, and the equation is solved for $(K L/r)$. The desired value for the critical stress is then obtained for this equivalent slenderness ratio from a column curve or table of column strengths for the material. A satisfactory column curve for aluminum alloy 178-T6 can be constructed on the basis of the yield strength of the material.[12] In

[12] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and M. Holt. *Technical Paper No. 1*, Aluminum Research Labs., New Kensington, Pa., 1938.

bending both tensile and compressive yielding occur. Compression tests were not made on the material of the beams tested in this investigation. However, from other tests on a similar product, it was determined that the average of compressive and tensile yield strengths will not vary by more than 5% from the tensile yield strength. The tensile yield strength values have been used in constructing the equivalent column curves in Fig. 5.

In Fig. 5(*a*) two branches of the column curve are shown for the region above elastic action because, as shown in Table 2, there was considerable variation in the yield strength of the material of the different channel beams. The upper branch corresponds to a yield strength value of 44,900 lb. per sq in., and the lower line represents a value of 39,800 lb per sq in. Arrows to the plotted points indicate the proper comparisons. The straight-line section of the equivalent column curve for the **Z**-beams (Fig. 5(*b*)) represents an average yield strength value of 36,600 lb per sq in.

The points in Fig. 5 represent the test results, expressed as modulus of failure values (M_r/Z), plotted against equivalent slenderness ratio values using for the fixity factor the values determined from the test results for the beams that buckled in the elastic range.

For both the channels (Fig. 5(*a*)) and the **Z**-beams (Fig. 5(*b*)), the beams that buckled in the plastic range failed at modulus of failure values higher than those indicated by the equivalent slenderness ratio method using straight-line column curves. This is consistent with results obtained from tests on **I**-beams.[3]

Design Considerations

When channels or **Z**-beams are used to carry concentrated or distributed loads on their flanges, nonuniform stress distributions will be present in the flanges unless restraints against lateral bending and rotation are introduced at points of loading and support. The design of **Z**-beams is generally complicated by the occurrence of unsymmetrical bending, resulting from the fact that the plane of loading does not generally contain a principal axis, and each case requires special consideration. In the instances in which an unsupported length of the **Z**-beam is subjected to pure bending, the lateral buckling problem can be solved as indicated.

It is not uncommon to have unsupported lengths of channels under pure bending in the direction of greatest stiffness. The solution to the lateral buckling problem is contained in Eq. 2. The only term in this equation not common to design engineers is the torsion-bending factor C_s. It would seem that the inclusion of this factor in the table of section properties of standard structural sections would be most useful. This factor is as much a property of the cross section as the moments of inertia and the torsion factor. In addition to appearing in equations for the lateral buckling of beams, this torsion-bending factor is essential in an analysis of the torsional buckling of columns and also enters into the determination of the strength and stiffness of a structural member subjected to torsion in such a manner that one or more of the cross sections are restrained against warping.[10]

Since values of C_s are not included in tables of section properties, the designer interested in investigating the lateral buckling of a channel beam may do one of three things:

1. Compute the value of C_s in accordance with Eq. 5, making approximate allowances for such flange tapers and fillets as may exist.
2. Evaluate C_s according to the approximate relationship,

$$C_s = \frac{I_f h^2}{2} \qquad (7)$$

3. Use the approximate relationship,

$$C_s = \frac{I_y h^2}{4} \qquad (8)$$

For symmetrical **I**-sections, Eqs. 7 and 8 are essentially the same. For channels, Eq. 7 will yield values for C_s smaller than the correct value and

Eq. 8 will give values larger than the correct value. The use of Eq. 7 for evaluating C_s is equivalent to the treatment given in a handbook[11] for the lateral buckling of channels. The use of Eq. 8 is equivalent to the design method included in the proposed specifications for aluminum alloy 14S-T6.[13]

[13] "Specifications for Heavy Duty Structure of High-Strength Aluminum Alloy: Progress Report of the Committee of the Structural Division on Design in Lightweight Structural Alloys," *Proceedings-Separate No. 22*, ASCE, June, 1950.

Computations for the torsion-bending factor in accordance with Eqs. 5, 7, and 8 have been made for the thirty-eight standard and fifteen special channels listed in a handbook.[11] Results of these computations indicate that C_s-values computed in accordance with Eq. 7 are low by from 25% to 58%, whereas with the use of Eq. 8 values for C_s result which are high by from 11% to 43%. Although these are serious errors in the evaluation of the torsion-bending factor, these errors have a greatly decreased effect in evaluating the lateral buckling strength. In Eq. 2, C_s enters only in the second term under the radical. This term becomes an important part of the total value under the radical when the ratio C_s/J is large, as in the case of channels with thin webs. The second term also increases in importance as the unsupported length ($K\,L$) becomes small.

In order to evaluate the effect on the critical buckling stress of errors resulting from the use of approximate expressions for obtaining C_s, computations have been made for the channel sections listed in Table 4 in which the approximate methods for evaluating C_s are in error as indicated. The term

TABLE 4.—THE EFFECT ON THE CRITICAL STRESS CAUSED BY AN ERROR IN THE TORSION-BENDING FACTOR

Channel elements	ERRORS IN C_s, in percents		ERRORS IN CRITICAL STRESS, in percents	
	Eq. 7	Eq. 8	Eq. 7	Eq. 8
3-in. by 0.170-in. web	−34	+21	− 3.4	+2.5
6-in. by 0.200-in. web	−41	+18	−12.5	+5.5
6-in. by 0.500-in. web	−49	+43	− 7	+5
12-in. by 0.300-in. web	−46	+15	−20	+6

in Eq. 2 containing C_s becomes of increasing importance as the unsupported length decreases. For small values of $K\,L$, buckling will involve stresses in the plastic ranges, and in the region of plastic buckling the critical moment becomes relatively insensitive to variations in C_s. The computations have been made for values of $K\,L$ which result in buckling stresses of approximately 50,000 lb per sq in.; this is considered a reasonable upper limit for elastic action in the high-strength structural aluminum alloys. The errors in buckling stress resulting from the use of the approximate values for C_s are given in Table 4. It can be seen that Eq. 7 results in a conservative estimate of the critical buckling stress. The use of Eq. 8 results in critical stress values which are higher than correct values but by not more than 6%.

It would appear, therefore, that the use of either Eq. 7 or Eq. 8 for approximating C_s could be permissible in designing channel beams for lateral buckling. The relative simplicity of Eq. 8 is evident since it contains terms

which can be taken directly from tables of section elements. This simplicity, coupled with the fact that the errors introduced are quite small, would seem to support a recommendation that until C_s-values are available for channel sections, Eq. 8 be used to approximate the value needed in computing the critical buckling stress for channel beams.

Substitution of Eq. 8 into Eq. 2 yields as an expression for the critical moment

$$M_c = \frac{\pi^2 E I_y h}{(K L)^2 2} \sqrt{1 + \frac{4}{\pi^2} \left(\frac{K L}{h} \right)^2 \frac{G J}{E I_y}} \quad \ldots\ldots\ldots\ldots\ldots (9)$$

Eq. 9 is applicable to both channels and I-beams.

Conclusions

The following conclusions have been drawn from the results and discussions of tests involving lateral buckling of aluminum alloy Z-beams and channels subjected to pure bending.

1. Considering the lack of complete lateral fixity of the ends of the unsupported lengths, the results of the tests on specimens that failed by elastic buckling are not incompatible with the theoretical solution obtained by Mr. Goodier.

2. For cases in which buckling occurred at stresses beyond the elastic range of the material, the equivalent slenderness ratio method, used with a straight-line column curve, was found to estimate the critical stress conservatively.

3. Values for the torsion-bending factor C_s should be included in tables of section elements for standard structural sections.

4. The design of Z-beams requires special attention for each application. Cases involving lateral buckling under pure bending applied in the plane of the web can be handled by the methods described.

5. For lateral buckling of channels under pure bending, a handbook design method[11] is conservative, probably by as much as 20% in extreme cases.

6. Until values for C_s are readily available, Eq. 9 should be used as a basis for the design of both I-beams and channels subjected to lateral buckling under pure bending. When applied to channels this method gives results which are high by 6% in extreme cases.

APPENDIX I. BENDING MOMENTS IN PRINCIPAL PLANES OF A Z-SECTION

The condition of uniform stress in the flanges obtained in the tests on the Z-beams indicated that there was no lateral deflection of the beam (excepting that associated with lateral buckling). This means that, in addition to the vertical bending moment applied in the plane of the web, there must have been a horizontal bending moment arising from the restraint of the end frame. Components of the vertical and horizontal bending moments can be resolved into the principal planes.

If M'_y equals the bending moment applied in the plane of the web and M'_x denotes the bending moment normal to the plane of the web,

$$M_y = M'_y \cos \alpha + M'_z \sin \alpha \quad \ldots\ldots (10)$$

and

$$M_z = M'_y \sin \alpha - M'_z \cos \alpha \quad \ldots\ldots (11)$$

The deflection δ in the direction of the y-axis and the z-axis can be expressed as

$$\delta_y = \frac{B}{I_z} (M'_y \cos \alpha + M'_z \sin \alpha) \quad \ldots\ldots (12)$$

and

$$\delta_z = \frac{B}{I_y} (M'_y \sin \alpha - M'_z \cos \alpha) \quad \ldots\ldots (13)$$

in which B is a coefficient involving the length, the conditions of loading and restraint, and the modulus of elasticity of the material.

The significant condition is that there is no horizontal deflection—that is, $\delta'_z = 0$; hence,

$$\delta'_z = \delta_y \sin\alpha - \delta_z \cos \alpha \quad \ldots\ldots (14)$$

By substituting Eqs. 12 and 13 into Eq. 14, and reducing,

$$M'_z (I_y \sin^2 \alpha + I_z \cos^2 \alpha) = M'_y (I_z - I_y) \sin\alpha \cos \alpha \quad \ldots\ldots (15)$$

By substitution of Eq. 15 into Eqs. 10 and 11, and simplifying,

$$M_y = \frac{M}{\cos \alpha + \dfrac{I_y \sin^2 \alpha}{I_z \cos \alpha}} \quad \ldots\ldots (16a)$$

$$M_z = \frac{M}{\sin \alpha + \dfrac{I_z \cos^2 \alpha}{I_y \sin \alpha}} \quad \ldots\ldots (16b)$$

in which $M = M'_y$ is the bending moment applied in the plane of the web.

By substituting for α, I_z, and I_y, the values given in Fig. 1 (*b*), and Table 1, Eqs. 16 reduce to

$$M_y = 1.079\ M \quad \ldots\ldots (3a)$$

and

$$M_z = 0.044\ M \quad \ldots\ldots (3b)$$

APPENDIX II. NOTATION

The following symbols, adopted for use in this paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Structural Analysis (ASA Z10.8-1949), prepared by a Committee of the American Standards Association, with Society representation and approved by the Association in 1949.

A = total area of the cross-sectional area:
- A_f = area of one flange;
- A_w = area of the web;

B = a coefficient;

C_s = torsion-bending factor;

E = modulus of elasticity of the material;
G = modulus of rigidity of the material;
h = height of the web;
I_f = moment of inertia of one flange about an axis through the centroid of the flange and parallel to the web:
- I_y = moment of inertia about the y-axis;
- I'_y = moment of inertia about the y'-axis;
- I_z = moment of inertia about the z-axis;
- I'_z = moment of inertia about the z'-axis;

J = torsion factor;
K = fixity coefficient for ends of unsupported spans;
L = unsupported span length;
M = bending moment applied in the plane of the web:
- M_c = critical bending moment;
- M_y = bending moment in the plane normal to the z-axis;
- M'_y = bending moment applied in the plane of the web;
- M_z = bending moment in the plane normal to the y-axis;
- M'_z = bending moment applied normal to the plane of the web;

r = radius of gyration of the section about an axis through the shear center and normal to the x-axis and y-axis;
Z = section modulus of the cross section;
α = angle between the principal plane of least stiffness and the plane of maximum restraint;
δ = deflection of the beam:
- δ_y = deflection of the beam in the direction of the y-axis;
- δ_z = deflection of the beam in the direction of the z-axis; and
- δ'_z = horizontal deflection of the beam.

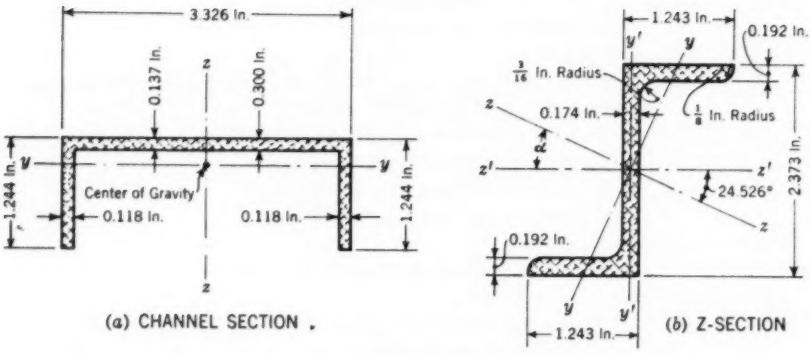

FIG. 1.—DIMENSIONS AND SECTION ELEMENTS

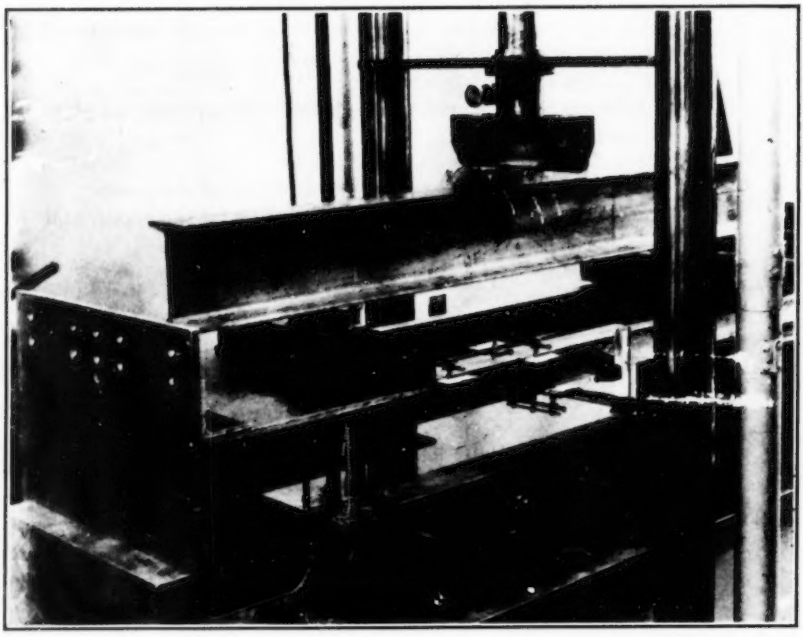

FIG. 2.—TEST ARRANGEMENT

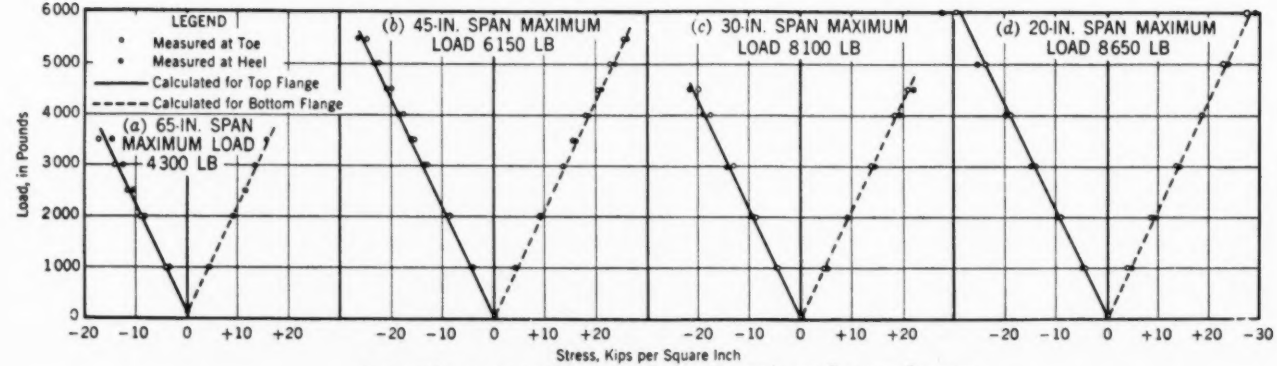
LEGEND
Measured at Toe
Measured at Heel
Calculated for Top Flange
Calculated for Bottom Flange
(a) 65-IN. SPAN MAXIMUM LOAD 4300 LB
(b) 45-IN. SPAN MAXIMUM LOAD 6150 LB
(c) 30-IN. SPAN MAXIMUM LOAD 8100 LB
(d) 20-IN. SPAN MAXIMUM LOAD 8650 LB
Load, in Pounds
Stress, Kips per Square Inch

FIG. 3.—COMPUTED AND MEASURED FLANGE STRESSES FOR CHANNEL SECTIONS

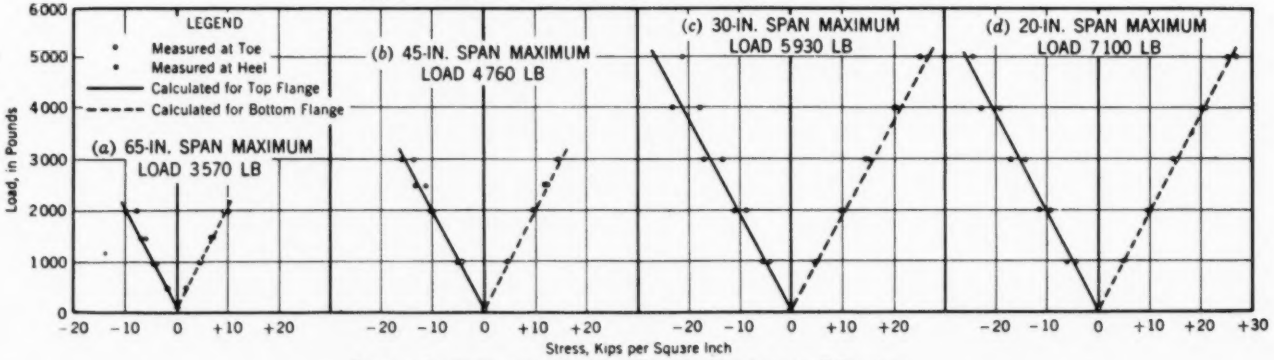
LEGEND
Measured at Toe
Measured at Heel
Calculated for Top Flange
Calculated for Bottom Flange
(a) 65-IN. SPAN MAXIMUM LOAD 3570 LB
(b) 45-IN. SPAN MAXIMUM LOAD 4760 LB
(c) 30-IN. SPAN MAXIMUM LOAD 5930 LB
(d) 20-IN. SPAN MAXIMUM LOAD 7100 LB
Load, in Pounds
Stress, Kips per Square Inch

FIG. 4.—COMPUTED AND MEASURED FLANGE STRESSES FOR Z-SECTIONS

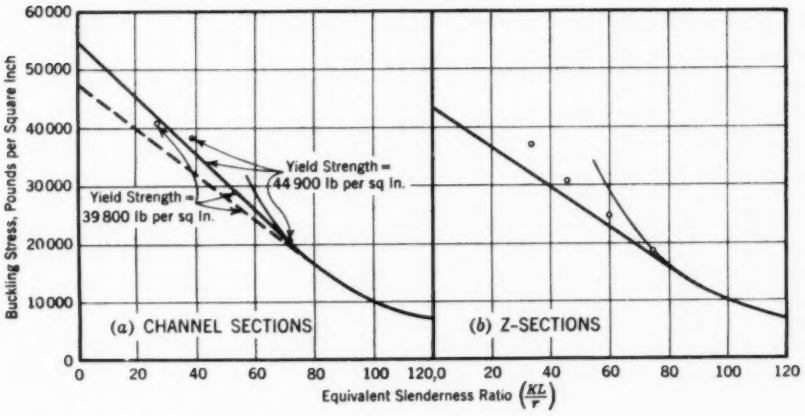

FIG. 5.—EQUIVALENT COLUMN CURVES